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An Exterior Solution of the Einstein Field Equations for a Rotating Infinite Cylinder

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Abstract

We give here a new exact solution to the exterior Einstein field equations for a rotating infinite cylinder. The solution is characterized by an everywhere singular metric. In the Papapetrou canonical coordinates, the 3-force acting on a radially moving test particle is

$$f^\alpha = \left(G \frac{m}{\sqrt{1 - v^2}} \frac{\lambda}{\rho}, 0, -\frac{m}{\sqrt{1 - v^2}} \frac{Cv}{\rho} \right)$$

where $\lambda > 0$. f^1 and f^3 are, respectively, the gravitational and Coriolis forces. The gravitational force is, therefore, repulsive.

1. *Introduction*

Solutions of Einstein field equations for axially symmetric sources have received considerable interests. A class of solutions for an uncharged rotating infinite cylinder has been obtained by Chakravarty (1974), Lewis (1932), and van Stockum (1937).

We give here a new, nontrivial solution to the latter problem. This solution is characterized by an everywhere singular metric. This property, on the one hand, explains why this solution was not obtained as a member of the Chakravarty's "exhusive" class of solutions. On the other hand, the singular character of the metric is presumably imposed by the physical problem at hand: The gravitational mass is infinite. Therefore, the distorsion in the geometry is likely to be infinite too. Furthermore, unlike the existing ones, our solution is constrained by a physical boundary condition at infinity.

Space-time singularities are of central interest in general relativity (Hawking and Ellis 1973). Here is an example where the space-time is everywhere singular. As a further interesting feature, in the Papapetrou canonical coordinates, the gravitational force turned out repulsive.

2. *The Field Equations*

Using the Papapetrou canonical coordinates (Papapetrou 1953), the line element reads

$$-ds^2 = f^{-1}\,[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2\, d\phi^2] - f(dt - \omega d\phi)^2 \qquad (2.1)$$

where, f, γ, and ω are functions of ρ and z only. One may use the exterior calculus technique, for example, to derive the Einstein field equations in vacuum (Ernst, 1967):

$$f\nabla^2 f = (\nabla f)^2 - \rho^{-2} f^4 (\nabla\omega)^2 \qquad (2.2)$$

$$\nabla\cdot(\rho^{-2} f^2 \nabla\omega) = 0 \qquad (2.3)$$

$$\gamma_{,\rho} = \tfrac{1}{4}\rho\,\{(1/f^2)(f_{,\rho}^2 - f_{,z}^2) - \rho^{-2} f^2 [(\omega_{,\rho})^2 - (\omega_{,z})^2]\} \qquad (2.4)$$

and

$$\gamma_{,z} = \tfrac{1}{2}\left\{\frac{f_{,\rho} f_{,z}}{f^2} - \rho^{-2} f^2 \omega_{,\rho}\omega_{,z}\right\} \qquad (2.5)$$

Here, ∇ is the ordinary gradient in the cylindrical coordinates: ρ, z, and ϕ.

3. *An Exact Solution Depending on ρ Only*

Consider an infinitely long cylindrical source. We *propose* the existence of a solution to equations (2.2)-(2.5) for the given problem where $f = \infty$ everywhere. However, the gradient ∂f is almost everywhere a nonsingular quantity. We prove this solution by self-consistency.

The singular character of the proposed solution makes the mathematics delicate. A concise way of arriving at such solution would be by determining its behavior for an arbitrarily long but finite cylindrical source. Then, letting the length approach infinity, we get the desired results. This is essentially our procedure. However, we make some arguments to simplify an otherwise very difficult problem.

Thus, let the source be of length L with ends at $z = \mp L/2$. In equations (2.2) and (2.3), as L gets arbitrarily large, the terms with partial derivatives with respect to z should get negligibly small compared to the *counterpart* terms with partial derivatives with respect to ρ. In the limit as $L \to \infty$, equations (2.2) and (2.3), respectively, reduce to

$$\lim_{L\to\infty} f\left(\frac{d^2 f}{d\rho^2} + \frac{1}{\rho}\frac{df}{d\rho}\right) = \left(\frac{df}{d\rho}\right)^2 - \rho^{-2} f^4 \left(\frac{d\omega}{d\rho}\right)^2 \qquad (3.1)$$

and

$$\lim_{L\to\infty} \rho^{-1} f^2 \frac{d\omega}{d\rho} = C_1 \qquad (3.2)$$

C_1 is an arbitrary constant. From equations (3.1) and (3.2) we find

$$\lim_{L\to\infty} f\left(\frac{d^2f}{d\rho^2} + \frac{1}{\rho}\frac{df}{d\rho}\right) = \left(\frac{df}{d\rho}\right)^2 - C_1^2 \tag{3.3}$$

We may restate our trial solution in term of the limit $L \to \infty$, as $\lim_{L\to\infty} f = \infty$ everywhere and $\lim_{L\to\infty} df/d\rho$ is almost everywhere a nonsingular quantity. This trial solution should imply, according to equation (3.3), that

$$\lim_{L\to\infty} \frac{d^2f}{d\rho^2} + \frac{1}{\rho}\frac{df}{d\rho} = 0 \tag{3.4}$$

Equation (3.4) has the first integral

$$\lim_{L\to\infty} \frac{df}{d\rho} = C_2/\rho \tag{3.5}$$

where C_2 is a second arbitrary constant.

The Boundary Condition. The existing solutions to the problem do not reduce to the flat space-time, nor do they have a *common* limiting form at infinity. The latter property leaves it open to determine the boundary condition on the metric tensor at infinity on a purely physical ground. A discussion of the boundary condition on the metric tensor at infinity is given by Petrov (1969). It is generally accepted that, in a suitable reference frame, the space-time metric tensor due to a finite, time-independent distribution of matter should reduce to the Minkowskian form at infinity: There is no gravitational radiation, while the space-time distortion due to the presence of matter should diminish at infinity. Accordingly, for the finite cylindrical source we can impose the boundary condition $\lim_{\rho\to\infty} f = 1$. This condition holds true for an arbitrarily long source. Therefore, it must be valid in the limit as $L \to \infty$. That is, $\lim_{L\to\infty} f(\rho = \infty) = 1$.

Using the above boundary condition together with $g_{00} = -f < 0$, we conclude $C_2 < 0$ and[1]

$$\lim_{L\to\infty} f(\rho) = \lim_{L\to\infty} f(\infty) - \lim_{L\to\infty} \int_{\rho}^{\infty} \frac{df}{d\rho}\, d\rho = \infty \tag{3.6}$$

Equations (3.5) and (3.6) show that our trial solution is *self-consistent.* That is, we have started by a trial solution of the form $\lim_{L\to\infty} f(\rho) = \infty$ everywhere and $\lim_{L\to\infty} df/d\rho$ is almost everywhere a nonsingular quantity. We substituted this trial solution into equation (3.3) and used a physical boundary condition, then there resulted an output solution, equations (3.5) and (3.6), that has the same input properties.

From equations (2.4), (2.5), (3.2), and (3.6) one finds

$$\lim_{L\to\infty} e^{2\gamma} = C_3 = \text{const} \tag{3.7}$$

1 Note that for an arbitrarily large ρ_0, we have

$$\lim_{L\to\infty} \int_{\rho}^{\rho_0} \frac{df}{d\rho}\, d\rho = \int_{\rho}^{\rho_0} \lim_{L\to\infty} \frac{df}{d\rho}\, d\rho = C_2 \ln(\rho_0) - C_2 \ln(\rho)$$

and

$$\lim_{L \to \infty} \omega = C_4 = \text{const} \tag{3.8}$$

From the spacelike nature of $d\rho$ and dz we should have, according to equation (2.1), $C_3 > 0$.

It should be mentioned here that although ω turned out constant in the limit $L \to \infty$, our solution may not be considered as a static type. We shall see in the next section that the derivative of ω, according to equation (3.2), gives rise to a Coriolis effect. Finally, owing to its everywhere singular character, the given solution is an entirely new one. It is not a special case of some existing solution, nor can it be implied by the general group theoretical considerations as given by Petrov (1969).

4. *The Equation of Motion*

We begin by defining the 3-velocity of a test particle as (Landau and Lifshitz, 1962)

$$v^\alpha = \frac{dx^\alpha}{\sqrt{h}(dx^0 - g_\alpha dx^\alpha)} \tag{4.1}$$

where, $h = -g_{00}$ and $g_\alpha = -g_{0\alpha}/g_{00}$. The 3-force, f^α, acting on a test particle of mass m in the gravitational field is then defined as

$$f^\alpha = \sqrt{1 - v^2}\, \frac{DP^\alpha}{ds} \tag{4.2}$$

where, D is a covariant differential relative to the 3-space metric tensor

$$\gamma_{\alpha\beta} = g_{\alpha\beta} - g_{0\alpha} g_{0\beta}/g_{00} \tag{4.3}$$

Furthermore,

$$P^\alpha = \frac{mv^\alpha}{\sqrt{1 - v^2}} \tag{4.4}$$

and

$$v^2 = \gamma_{\alpha\beta} v^\alpha v^\beta \tag{4.5}$$

It follows from equation (4.2) that

$$f^\alpha = \frac{m}{\sqrt{1 - v^2}} \gamma^{\alpha\mu} \left[-\frac{\partial}{\partial x^\mu} (\ln\sqrt{h}) + \sqrt{h} \left(\frac{\partial g_\nu}{\partial x^\mu} - \frac{\partial g_\mu}{\partial x^\nu} \right) v^\nu \right] \tag{4.6}$$

We apply equation (4.6) to the space-time metric tensor found in the previous section. We must, however, first carry the computations for a *finite* length source. By letting the length approach infinity, we then get the appropriate force.

Thus, consider a radially moving test particle. From equations (2.1) and (4.1) one finds

$$v^{\alpha} = (v\sqrt{f}e^{-\gamma}, 0, 0) \tag{4.7}$$

Substituting v^{α} and the metric tensor of equation (4.7) and (2.1), respectively, into equation (4.6), *then* letting L approach infinity and finally noting equations (3.2) and (3.5), we find

$$f^{\alpha} = \left(G\frac{m}{\sqrt{1-v^2}}\frac{\lambda}{\rho}, 0, -\frac{m}{\sqrt{1-v^2}}\frac{Cv}{\rho}\right) \tag{4.8}$$

where, $G\lambda = -C_2/2C_3$ and $C = C_1/\sqrt{C_3}$. It should be remarked that f^{α} *is a well-defined quantity.*

From the conditions found previously, $C_2 < 0$ and $C_3 > 0$,[2] we find $\lambda > 0$. Thus, f^1, the gravitational force, is *repulsive.* For a moving test particle, f^3 represents the Coriolis effect.

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2 These conditions were required for proper metric signature.